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On Modular Invariant Partition Functions of Conformal Field Theories with Logarithmic Operators

Michael A.I. Flohr*
Instituto de Matemáticas y Física Fundamental, C.S.I.C., Serrano 123, E-28006 Madrid, Spain

Abstract

We extend the definitions of characters and partition functions to the case of conformal field theories which contain operators with logarithmic correlation functions. As an example we consider the theories with central charge $c = c_{p,1} = 13 - 6(p + p^{-1})$, the "border" of the discrete minimal series. We show that there is a slightly generalized form of the property of *rationality* for such logarithmic theories. In particular, we obtain a classification of theories with $c = c_{p,1}$ which is similar to the *A*-*D*-*E* classification of $c = 1$ models.

*email: `ifflohr@roca.csic.es`

1 Introduction

This paper mainly deals with conformal field theories (CFT) with central charge $c = c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$, $p \in \mathbb{N}$, i.e. we consider the "boundary" of the minimal discrete series.

One particularly interesting CFT is the theory with central charge $c = -2$. It appears in the theoretical treatment of two-dimensional polymers and self avoiding walks [DuSa87, Sal92], in the quantum Hall effect [W^{2}H94], and in the phenomenon of unifying $\mathcal{W}$-algebras [BEH^{3}94]. It is the first member of a unique series of extended conformal algebras generated by a triplet of primary fields [Kau91], which have central charge $c = c_{p,1}$, $p \in \mathbb{N}$, and – last but not least – it is the simplest example of a theory containing logarithmic operators [Gur93].

Nonetheless, not very much is known on the representation theory of this CFT or the other members of the $c_{p,1}$ series. Moreover, while it has been explicitly shown for $p = 2, 3$ that there are only finitely many irreducible highest weight representations with respect to the maximally extended chiral symmetry algebra [EFH^{2}V93, EH^{2}93], the characters and partition functions could not been derived yet.

The aim of this letter is to extend the definitions of characters and partition functions such that they cover the case of logarithmic operators, which necessarily appear in a CFT, whenever the differential equations arising from the conformal Ward identities and the existence of singular vectors yield degenerate solutions [Gur93]. With these improved definitions we can calculate the characters for all irreducible highest weight Jordan cell representations of the $c_{p,1}$-series. We then write down the modular invariant partition functions of these models and show that logarithmic theories may be rational, if one slightly weakens the standard presumptions for rationality of a CFT. This leads to a new class of partition functions of CFTs with $c_{\mathit{eff}} = 1$, which resemble an *A-D-E* classification similar to the well known $c = 1$ models. Next, we show that the Verlinde formula for the calculation of the fusion coefficients from the S-matrix cannot be longer valid and discuss some possible generalizations. Finally, we discuss the consequences of our results for $c = -2$ to the theory of polymers, where an additional structure shows up.

We concentrate ourselves on the case of the $c_{p,1}$ models, since these are the only ones, where the appearance and behavior of logarithmic operators is well understood. To fix notation, we shortly review these models here:

1.1 The $c_{p,1}$ models

In a work of H.G. Kausch [Kau91] the possibility to extend the Virasoro algebra by a multiplet of fields of equal conformal dimension has been considered. Besides some sporadic solutions he found a series of algebras extended by a singlet or triplet of fields of odd dimension which resemble a SO(3) structure. The operator product expansion (OPE) is given by

$$W^{(j)}(z)W^{(k)}(\zeta) = \frac{c}{\Delta}\delta^{jk}\frac{1}{(z-\zeta)^{2\Delta}} + C_{\Delta\Delta\Delta}i\varepsilon^{jkl}\frac{W^{(l)}(\zeta)}{(z-\zeta)^{\Delta}} + \mathit{descendant\ fields}\,, \tag{1.1}$$

where $c = c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$ and $\Delta = 2p - 1$. These CFT posses infinitely many degenerate representations with integer conformal weights

$$h_{2k+1,1} = k^2p + kp - k\,. \tag{1.2}$$

These representations correspond to a set of relatively local chiral vertex operators. But there is a peculiarity: The energy operator L_0 is no longer diagonal on these degenerate representations, but is given in a Jordan normal form with non-trivial blocks.

A standard free field construction [BPZ83, DoFa84] shows that the degenerate fields have conformal weights $h_{m,n} = \frac{\alpha_{m,n}^2}{4} + \frac{c_{p,1}-1}{24}$, where $\alpha_{m,n} = m\sqrt{p} - n\sqrt{p}^{-1}$. The fundamental region of the minimal models unfortunately is empty: $\{m, n | 1 \leq m < 1, 1 \leq n < p\} = \emptyset$. But without loss of generality we can reduce the labels (m, n) to the region $0 < m, 0 < n \leq p$. Moreover, we have the following abstract fusion rules which result from the conditions for the existence of well defined chiral vertex operators [Kau9?].

Proposition 1. *Let $c = 13 - 6(p + p^{-1})$ with $p \in \mathbb{N}$. Then there exist well defined chiral vertex operators for triples of Virasoro highest weight representations to $(h_{m_1,n_1}, h_{m_2,n_2}, h_{m_3,n_3})$ with $0 < m_i$ and $0 < n_i \leq p$ iff $|m_1 - m_2| < m_3 < m_1 + m_2$ and $|n_1 - n_2| < n_3 \leq \min(p, n_1 + n_2 - 1)$, and moreover $m_1 + m_2 + m_3 - 1 \equiv n_1 + n_2 + n_3 - 1 \equiv 0 \bmod 2$.*

The screening charges have a special meaning. With $\alpha_\pm = \alpha_0 \pm \sqrt{1 + \alpha_0^2}$ and $\alpha_0^2 = (1-p)^2/4p$ the first of them is given by

$$Q = \int_{\Omega_1} \frac{dz}{2\pi i} V_{\alpha_+}(z)\,,$$

where Ω_1 encircles the origin counterclockwise in the standard way. Q has trivial monodromy on the Fock spaces $\mathfrak{F}_{m,n}$ of the free field construction on the weights $h_{m,n}$, and therefore is by itself a well defined local chiral vertex operator $Q : \mathfrak{F}_{m,n} \to \mathfrak{F}_{m-2,n}$. This screening charge is exactly responsible for the multiplet structure of the chiral fields. We have $Q^m = 0$ on $\mathfrak{F}_{m,n}$. The other screening charge (to the "power" k) is

$$\tilde{Q}^k = \int_{\Omega_k} \frac{dz_1}{2\pi i} \cdots \frac{dz_k}{2\pi i} V_{\alpha_-}(z_1) \ldots V_{\alpha_-}(z_k)\,,$$

where the integration path is radially ordered, $|z_1| > \ldots > |z_k|$, and encircles the origin. It is well defined on $\mathfrak{F}_{m,n}$ iff $0 < k = n < p$. $\tilde{Q}^p$ vanishes identically on $\mathfrak{F}_{m,p}$. The BRST-identity is $\tilde{Q}^{p-n}\tilde{Q}^n = 0$, such that we have the following embedding structure of Fock spaces (see [Fel89, F^{2}K89]) induced by the exact sequence

$$\ldots \xrightarrow{\tilde{Q}^{p-n}} \mathfrak{F}_{m-2,n} \xrightarrow{\tilde{Q}^{n}} \mathfrak{F}_{m-1,p-n} \xrightarrow{\tilde{Q}^{p-n}} \mathfrak{F}_{m,n} \xrightarrow{\tilde{Q}^{n}} \mathfrak{F}_{m+1,p-n} \xrightarrow{\tilde{Q}^{p-n}} \mathfrak{F}_{m+2,n} \xrightarrow{\tilde{Q}^{n}} \ldots$$

The Virasoro modules are then given by $\mathfrak{H}_{m,n} = \ker_{\mathfrak{F}_{m,n}} \tilde{Q}^n$. The fields $\phi_{2k+1,1} \equiv V_{\alpha_{2k+1,1}}$, $k \in \mathbb{N}$, all have integer dimensions $h_{2k+1,1} = k^2 p + kp - k$, such that one is tempted to extend the local chiral algebra by them. Indeed, from proposition 1 follows that the local chiral algebra generated by only the stress energy tensor and the field $\phi_{3,1}$ closes, since no other fields can contribute to the singular part of the OPE. The multiplet structure is obtained by repeated application of Q, $W^{(j)} = Q^j \phi_{3,1}$. Indeed, this yields three fields with SO(3)-structure [Kau91], and therefore a $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$-algebra. With $W = \sum_j W^{(j)}$ we get the symmetric singlet algebra $\mathcal{W}(2, 2p-1)$.

With the BRST-structure given above one can construct exactly $2p$ (regular) representations of the fully extended chiral algebra by taking into account the multiplets generated

by the Q-operator[i]. Formally we can write these $\mathcal{W}$-modules as

$$\mathfrak{H}^{\mathcal{W}}_{n,+} = \bigoplus_{j=0}^{\infty} \bigoplus_{m=0}^{2j-1} Q^m \mathfrak{H}_{2j+1,n}\,, \tag{1.3}$$

$$\mathfrak{H}^{\mathcal{W}}_{n,-} = \bigoplus_{j=1}^{\infty} \bigoplus_{m=0}^{2j-2} Q^m \mathfrak{H}_{2j,n}\,, \tag{1.4}$$

with $1 \leq n \leq p$. The corresponding conformal weights are $h_{1,n}$ and $h_{2,n}$ respectively. The $\mathcal{W}$-representations for $h_{1,n}$ are singlets, the ones for $h_{2,n}$ doublets. There also exist special representations for the weights $h_{0,n}, 1 \leq n < p$. Their highest weight vectors are singular vectors in $\mathfrak{F}_{1,p-n}$, which have the *same* highest weights. The corresponding chiral vertex operators are degenerated. For instance, there are besides the identity $p-1$ additional vertex operators of conformal weight zero, which map $\mathfrak{F}_{0,n}$ to $\mathfrak{F}_{1,p-1}$. Consequently, also the descendant fields of the identity family are degenerated, in particular the Virasoro field itself. This forces the existence of non-trivial Jordan cells for L_0, i.e. L_0 no longer is diagonalizable. Moreover, the multiplicities of states in the Virasoro modules must change. We have, in sloppy terms, a p-fold degenerate identity, which will lead to a multiplicity of p in the characters of the highest weight representations $h_{0,n}$.

1.2 Structure constants

Although the chiral vertex operator algebra is degenerated, it is possible to explicitly calculate their structure constants with the methods given in [F^{2}K89, Flo93]. First, we make the usual ansatz for the decomposition of local chiral fields into chiral vertex operators,

$$W^{(n,n')}(z) = \sum_{l,l',m,m'} \mathcal{D}^{(l,l')}_{(n,n')(m,m')} V^{h_{m,m'}}_{h_{l,l'}h_{n,n'}}(\cdot, z)\,, \tag{1.5}$$

where the chiral vertex operators are maps $\mathfrak{H}_{h_{n,n'}} \mapsto \mathrm{Hom}(\mathfrak{H}_{h_{m,m'}}, \mathfrak{H}_{h_{l,l'}})$. In our case we have $n' = m' = l' = 1$ and n, m, l odd, where $h_{2k+1,1}$ is given by (1.2). The coefficients $\mathcal{D}^{(l,1)}_{(n,1)(m,1)}$ important for us are given by

$$\begin{aligned}
\left(\mathcal{D}^{(l,1)}_{(n,1)(m,1)}\right)^2 &= c \cdot \frac{h_{l,1}}{h_{n,1}h_{m,1}} \frac{N^{(1,1)}_{(l,1)(l,1)}}{N^{(1,1)}_{(n,1)(n,1)} N^{(1,1)}_{(m,1)(m,1)}} \left(\Delta^l_{n,m}(x)\Delta^1_{1,1}(x')\right)^2\,, &(1.6)\\
\Delta^l_{n,m}(x) &= (-1)^{\frac{1}{2}(n+m-l-1)} \left(\frac{[n]_x[m]_x[l]_x}{[1]_x}\right)^{\frac{1}{2}}\\
&\times \prod_{j=(l+n-m+1)/2}^{n-1} [j]_x \prod_{j=(m+n-l+1)/2}^{n-1} [j]_x \prod_{j=(l+m-n+1)/2}^{(l+m+n-1)/2} \frac{1}{[j]_x}\,. &(1.7)
\end{aligned}$$

Here, we have defined $[j]_x = x^{j/2} - x^{-j/2}$ with $x = \exp(2\pi i p)$ and $x' = \exp(2\pi i p^{-1})$. The prefactor $c \cdot h_{l,1}h^{-1}_{n,1}h^{-1}_{m,1}$ stems from our normalization of the two-point-functions, which we have chosen for simple primary fields to be

$$\langle 0|W^{(n,1)}_{-h_{n,1}} W^{(m,1)}_{h_{m,1}}|0\rangle = \frac{c}{h_{n,1}}\delta_{n,m}\,. \tag{1.8}$$

[i]The operators Q and $\tilde{Q}^k$ generate four two-dimensional complexes of the $\mathfrak{F}_{m,n}$, one for m even and odd respectively, and one for $n = p$ and $n \neq p$ respectively [Kau9?].

The normalization integrals $N_{(n,1)(m,1)}^{(l,1)}$, i.e. the three-point-functions of the chiral vertex operators, can be derived from the general solution of the Fuchsian differential equations for degenerate representations of the Virasoro algebra [DoFa84]. They are particularly simple in our case. With $r = \frac{1}{2}(n+m-l-1)$ denoting the number of inserted screening charges they read

$$\begin{aligned} N_{(n,1)(m,1)}^{(l,1)} &= (-1)^{\frac{1}{2}r} \prod_{j=1}^{r} \frac{[m-j]_x [j]_x}{[1]_x} \\ &\times \prod_{j=1}^{r} \frac{\Gamma(jp)\Gamma(1+(j-m)p)\Gamma(1+(j-n)p)}{\Gamma(p)\Gamma(2+(r-m-n+j)p)} \,. \end{aligned} \tag{1.9}$$

The structure constants of the OPE or equivalently of the Lie algebra of the Fourier modes of the local chiral fields are then

$$C_{(n,1)(m,1)}^{(l,1)} = \mathcal{D}_{(n,1)(m,1)}^{(l,1)} N_{(n,1)(m,1)}^{(l,1)} \,. \tag{1.10}$$

Let us introduce the following symmetry factor $\mathcal{S}(n,m,l)$, since the extended symmetry algebra, due to its $\mathfrak{su}(2)$-structure, contains either a multiplet of fields of the same dimension or the symmetric singlet of the latter. The reason for this is that the screening charge itself is a local operator acting on local fields, as long as $p \in \mathbb{N}$. In our case we have multiplets $W_l^{(j)} = Q^j W^{(l,1)}$, $0 \leq j < l$, and their symmetric singlets $W_l = \sum_{j=0}^{l-1} W_l^{(j)}$. From the formal fusion rules for degenerated representations

$$W_l^{(j)} \star W_{l'}^{(j')} = \sum_{\substack{m=|l-l'|+1 \\ l+l'-m-1 \equiv 0 \bmod 2}}^{l+l'-1} W_m^{(j+j'-r)} \,, \tag{1.11}$$

where again $r = \frac{1}{2}(l+l'-m-1)$, we can read off the multiplicities for the formal fusion rules of the singlets,

$$W_l \star W_{l'} = \sum_{\substack{m=|l-l'|+1 \\ l+l'-m-1 \equiv 0 \bmod 2}}^{l+l'-1} \mathcal{S}(l,l',m) W_m = \sum_{\substack{m=|l-l'|+1 \\ l+l'-m-1 \equiv 0 \bmod 2}}^{l+l'-1} \begin{pmatrix} \frac{1}{2}(l+l'-|l-l'|-2) \\ \frac{1}{2}(m-|l-l'|-1) \end{pmatrix} W_m \,, \tag{1.12}$$

which essentially are the multiplicities arising in tensoring symmetric $\mathfrak{su}(2)$ Young tableaus. Next, we determine the phases $\Delta_{n,m}^l(x)$. Trivially, the phase $\Delta_{1,1}^1(x') = 1$ for all x'. Since $p \in \mathbb{N}$, we find that $[k]_x = 0$. Actually, considering $[k]_x$ for $p+\varepsilon$ in the limit $\varepsilon \to 0$, we get to leading order of ε the expression

$$[k]_x = (-1)^{kp} 2\pi i k \varepsilon \quad (\varepsilon \to 0) \tag{1.13}$$

for $p \in \mathbb{N}$. All these $2\pi i \varepsilon$ factors exactly cancel in numerator and denominator such that $\Delta_{n,m}^l(x)$ is analytic also for integer p. Naively, one would expect that the result is just a sign, since all considered operators are local to each other. Instead of this we find that $\left(\Delta_{n,m}^l(x)\right)^2$ are rational numbers, which stem from the multiplet structure,

$$\begin{aligned} \Delta_{n,m}^l(x) &= (-1)^{lp} (-1)^{\frac{1}{2}(n+m-l-1)(p+1)} (-1)^{p((lm+ln+nm)/2-(l^2+m^2+n^2-1)/4)} \\ &\times \frac{\sqrt{nml}(n-1)!^2(\frac{1}{2}(l+m-n-1))!}{(\frac{1}{2}(l+n-m-1))!(\frac{1}{2}(m+n-l-1))!(\frac{1}{2}(l+m+n-1))!} \,. \end{aligned} \tag{1.14}$$

In the same way we get singularities in the normalization integrals for $p \in \mathbb{N}$. Nonetheless, the square of $C^{(l,1)}_{(n,1)(m,1)}$ is still well defined and finite, since all singularities cancel. Moreover, the values for integer p are precisely given as the analytical continuation of the region $p \in \mathbb{R}_+ - \mathbb{N}$: In the limit $\varepsilon \to 0$ we get for $m \in \mathbb{Z}$, $p \in \mathbb{N}$ to leading order in ε

$$\Gamma(m(p+\varepsilon)) = \begin{cases} (mp-1)! & mp > 0\,, \\ \frac{(-1)^{mp}}{(-mp)!}\frac{1}{m\varepsilon} & mp \leq 0\,. \end{cases} \tag{1.15}$$

Finally, plugging this in the normalization integrals, we arrive at the result

$$\begin{aligned} \left(C^{(l,1)}_{(n,1)(m,1)}\right)^2 &= \frac{c}{\mathcal{S}(n,m,l)}\frac{h_{l,1}}{h_{m,1}h_{n,1}}(\varphi(\Delta^l_{n,m}))^2 \prod_{j=1}^{r} \frac{((pj-1)!^2(p(r+l+1-j)-2)!^2}{p(m-j)-1)!^2(p(n-j)-1)!^2} \\ &\times \prod_{j=1}^{n-1}\frac{(p(n-j)-1)!^2}{(pj-1)!(p(n-j+1)-2)!}\prod_{j=1}^{m-1}\frac{(p(m-j)-1)!^2}{(pj-1)!(p(m-j+1)-2)!} \\ &\times \prod_{j=1}^{l-1}\frac{(pj-1)!(p(l-j+1)-2)!}{(p(l-j)-1)!^2}\,, \end{aligned} \tag{1.16}$$

where again $r = \frac{1}{2}(n+m-l-1)$. We denote by $\varphi(\Delta^l_{n,m})$ the phase part (a sign) of $\Delta^l_{n,m}$ from (1.14), since the modulus combines nicely with corresponding terms from the normalization integrals to $\mathcal{S}(n,m,l)^{-1}$.

We may check these formulas with already known results. Firstly, we obtain the following explicit expression for the self-coupling constant $C^{\Delta}_{\Delta\Delta}$ in (1.1),

$$\left(C^{(3,1)}_{(3,1)(3,1)}\right)^2 = c_{p,1}(-1)^p\frac{(4p-2)!^2(p-1)!^3}{2(3p-2)!(2p-1)!^4}\,. \tag{1.17}$$

Here, the $\mathfrak{su}(2)$ multiplicity of the triplet on the rhs of a symmetric tensor product of two triplets is $\mathcal{S}(3,3,3) = 2$. The expression (1.17) is precisely the one earlier obtained by H.G. Kausch [Kau91] by explicitly integrating the screening charges. Since the field W_5 has even dimension $h_{5,1} = 6p-2$, its OPE with itself has no term proportional to W_3 with the odd dimension $h_{3,1} = 2p-1$. Therefore, we may construct a non-trivial, even sector subalgebra $\mathcal{W}(2, 6p-2)$. In fact, the next field of even dimension, W_9, has $h_{9,1} = 20p-4 > 2h_{5,1}-1$, and thus does not appear on the rhs of the OPE. Since $h_{5,1}$ is even, the self coupling does not vanish even in the singlet case. Two examples of this subalgebra series could be constructed explicitly, namely $\mathcal{W}(2,4)$ at $c = 1$ [BFKNRV91, KaWa91] and $\mathcal{W}(2,10)$ at $c = -2$ [EH²93]. The self-coupling constants are $\frac{50}{3}$ and $-\frac{352836}{5}$ respectively. Our explicit formula (1.16) yields with $\mathcal{S}(5,5,5) = 6$ the expression

$$\left(C^{(5,1)}_{(5,1)(5,1)}\right)^2 = c_{p,1}(-1)^{p-1}\frac{(2p-1)!^3(p-1)!^3(7p-2)!^2(6p-2)!(6p-3)!}{6(4p-1)!^3(3p-1)!^3(5p-2)!(4p-2)!(3p-2)!(2p-2)!}\,, \tag{1.18}$$

which for $p = 1$ and $p = 2$ gives the desired numbers.

2 Representations and Characters

Let us assume that the Hilbert space $\mathfrak{H} \otimes \bar{\mathfrak{H}}$ is a direct sum of irreducible highest weight representations (HWR) with respect to the chiral symmetry algebra $\mathcal{W}$,

$$\mathfrak{H} \otimes \bar{\mathfrak{H}} = \bigoplus_{\lambda\in\Lambda}\mathfrak{H}^{(\lambda)} \otimes \bigoplus_{\bar{\lambda}\in\bar{\Lambda}}\bar{\mathfrak{H}}^{(\bar{\lambda})}\,. \tag{2.1}$$

Further we assume that $\mathcal{W}$ is maximal such that $\Lambda = \bar{\Lambda}$ is the set of all $\mathcal{W}$ HWRs, i.e. the theory is *symmetric*. We decompose $\mathfrak{H}^{(\lambda)}$ into Virasoro HWRs, the set of them we denote with N_λ,

$$\mathfrak{H} \otimes \bar{\mathfrak{H}} = \bigoplus_{\lambda\in\Lambda} \left(\bigoplus_{\nu\in N_\lambda} \mathfrak{H}_\nu^{(\lambda)} \otimes \bigoplus_{\nu\in N_\lambda} \bar{\mathfrak{H}}_\nu^{(\lambda)} \right) . \tag{2.2}$$

A CFT is said to be *rational*, iff $|\Lambda| < \infty$. It is called *quasi-rational*, if Λ is countable. The Cartan subalgebra $\mathcal{C}$ is spanned by L_0, the central extension C and the zero modes of the simple primary fields $\phi_i \in \mathcal{B}_\mathcal{W}$ which generate the $\mathcal{W}$-algebra. We denote the highest weight state (HWS) of $\mathfrak{H}_\nu^{(\lambda)}$ by $|\mathbf{h}^{(\lambda)}\rangle = |c, h_\nu, w_1^{(\lambda)}, w_2^{(\lambda)}, \ldots\rangle$ where $\mathbf{h} \in \mathcal{C}^*$ is the highest weight vector (HWV). A regular HWR $M_{|\mathbf{h}\rangle}$ of a $\mathcal{W}$-algebra to a HWS $|\mathbf{h}\rangle = |c, h, w_1, w_2, \ldots\rangle$ is then defined to satisfy the following conditions:

$$\begin{aligned}
C|\mathbf{h}\rangle &= c|\mathbf{h}\rangle\,, \\
L_0|\mathbf{h}\rangle &= h|\mathbf{h}\rangle\,, \\
\phi_{i,0}|\mathbf{h}\rangle &= w_i|\mathbf{h}\rangle \ \ \forall \phi_i \in \mathcal{B}_\mathcal{W}\,, \\
L_n|\mathbf{h}\rangle &= 0 \ \ \forall n > 0\,, \\
\phi_{i,n}|\mathbf{h}\rangle &= 0 \ \ \forall \phi_i \in \mathcal{B}_\mathcal{W} \text{ and } \forall n > 0\,, \\
M_{|\mathbf{h}\rangle} &= U(\mathcal{W})|\mathbf{h}\rangle\,,
\end{aligned}$$

where $U(\mathcal{W})$ denotes the universal enveloping algebra of $\mathcal{W}$. Moreover, we call a HWR $V_{|\mathbf{h}\rangle}$ *Verma module*, iff the sequence

$$V_{|\mathbf{h}\rangle} \longrightarrow M_{|\mathbf{h}\rangle} \longrightarrow 0 \tag{2.3}$$

is exact for all HWRs $M_{|\mathbf{h}\rangle}$. The Verma module $V_{|\mathbf{h}\rangle}$ has a natural gradation

$$V_{|\mathbf{h}\rangle} = \bigoplus_{n\in\mathbb{Z}_+} V_{|\mathbf{h}\rangle}^n\,, \tag{2.4}$$

where $V_{|\mathbf{h}\rangle}^n$ is the L_0 eigenspace with eigenvalue $h + n$.

Let us now assume that there exist HWRs, whose L_0 eigenvalues differ by integers. We must distinguish two cases. If the difference Δh of the L_0 eigenvalues of two HWRs is always non zero, or the HWVs differ in at least one component, it still is possible to diagonalize L_0, even if $\Delta h \in \mathbb{Z}$. Moreover, there are no logarithmic operators necessary. The reason is that the differential equations for the conformal Ward identities do not degenerate in this case. This is different to the case of the modular differential equation to be satisfied by the characters, which is only sensible modulo integers. Examples of such rational CFTs with HWRs with $\Delta h \in \mathbb{Z}$ can be found in [Flo93, Flo94].

Therefore, we now assume the existence of $n+1 > 1$ HWRs such that $\mathbf{h}_i - \mathbf{h}_j = 0$ for $1 \le i, j \le n+1$. We call such CFTs *logarithmic*. As V. Gurarie [Gur93] pointed out, we have to modify the definition of HWRs in the following way: The HWS is replaced by a non-trivial Jordan cell of L_0 of dimension $n+1$, which is spanned by $\{|\mathbf{h}; 0\rangle = |\mathbf{h}\rangle, |\mathbf{h}; 1\rangle, \ldots, |\mathbf{h}; n\rangle\}$. We then will call $M(|\mathbf{h}; \mathbf{m}\rangle)_{0\le m\le n}$ a *logarithmic* HWR of a $\mathcal{W}$-algebra to the highest weight L_0-Jordan cell of rank $n+1$, $(|\mathbf{h}; \mathbf{m}\rangle = |c, h, w_1, w_2, \ldots; m\rangle)_{0\le m\le n}$, if it satisfies the following conditions:

$$\begin{aligned}
L_0|\mathbf{h}; m\rangle &= h|\mathbf{h}; m\rangle + |\mathbf{h}; m-1\rangle\,, \quad m > 0\,, \\
L_0|\mathbf{h}; 0\rangle &= h|\mathbf{h}; 0\rangle\,, \\
\phi_{i,0}|\mathbf{h}; m\rangle &= w_i|\mathbf{h}; m\rangle + \ldots\,, \quad m > 0\,, \ \ \forall \phi_i \in \mathcal{B}_\mathcal{W}\,,
\end{aligned} \tag{2.5}$$

and otherwise the conditions of the original definition. The dots in the last condition represent possible non-diagonal contributions. In addition, there is in general no orthogonal system of states within the Jordan cell, i.e. $\langle\mathbf{h};k|\mathbf{h};l\rangle \neq 0$ even for $k \neq l$.

Since the other properties of HWRs remain unchanged, it makes sense to consider such logarithmic HWRs if the whole Jordan cell structure is taken into account for the definition of $\mathcal{W}$-families.

Next, we want to discuss the consequences for the characters. For simplicity, we consider a Jordan cell of form $\binom{h\ 1}{0\ h}$, i.e. we have two HWSs, $|h;0\rangle$ and $|h;1\rangle$, on which the action of L_0 is given by $L_0|h;0\rangle = h|h;0\rangle$ and $L_0|h;1\rangle = h|h;1\rangle + |h;0\rangle$. The off-diagonal element could be any non-zero number, since a Jordan cell decomposition is just one particular choice. The physical correct decomposition will be fixed later by modular invariance.

The HWS $|h;0\rangle$ is an ordinary L_0-eigenstate, such that the character of the corresponding HWR should be defined in the usual manner. The other state, $|h;1\rangle$ is not a L_0-eigenstate, application of L_0 generates a new state, which is not contained in the standard Verma module. If we apply L_0 once again, this state is recovered plus an additional one, etc. Thus, the operator L_0, acting on the Jordan cell, may be written as $L_0 = \binom{L_{0;0}\ \ 1}{0\ \ L_{0;1}}$, where the second label j refers to the Verma like modules on which the $L_{0;j}$ operators act.

The character of a HWR on a HWS $|\mathbf{h}\rangle$ is usually defined as

$$\chi_{|\mathbf{h}\rangle}(q) = \mathrm{tr}_{M_{|\mathbf{h}\rangle}} q^{L_0 - c/24}\,, \tag{2.6}$$

where $q = \exp(2\pi i\tau)$ is up to now a formal variable, and the trace is taken over the module which is created by action of $U(\mathcal{W})$ on $|\mathbf{h}\rangle$. Using our L_0 matrix, we obtain

$$\begin{aligned} q^{L_0} &= \sum_{n=0}^{\infty} \frac{(2\pi i\tau)^n}{n!} \left(\begin{array}{cc} L_{0;0} & 1 \\ 0 & L_{0;1} \end{array}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(2\pi i\tau)^n}{n!} \left(\begin{array}{cc} L_{0;0}^n & nL_{0;0}^{n-1} \\ 0 & L_{0;1}^n \end{array}\right)^n \\ &= \left(\begin{array}{cc} q^{L_{0;0}} & 2\pi i\tau q^{L_{0;0}} \\ 0 & q^{L_{0;1}} \end{array}\right)\,. \end{aligned} \tag{2.7}$$

Since formally $2\pi i\tau = \log(q)$, we see that a non-trivial Jordan cell may generate logarithmic terms in the character expansions. This is completely analogous to the logarithms in the correlation functions of certain operators, which stem from the degeneracy of the conformal Ward identity differential equations: We obtain essentially the same degeneracies in the modular differential equations for the characters, which force additional solutions with logarithms.

The careful reader may wonder, how the logarithmic terms can show up in the characters. Usually, traces (2.6) over modules are well defined, since the complete Hilbert space is a direct sum of modules and L_0 can be uniquely restricted to one of the modules. Now, if L_0 has non trivial Jordan form, modules $M_{|\mathbf{h};k\rangle}$ and $M_{|\mathbf{h};l\rangle}$ are not orthogonal. Therefore, the characters depend on the choice of a basis of generating states, while the sum $\sum_{k=0}^{n} \chi_{|\mathbf{h};k\rangle}(q)$ is invariant under any basis change $|\tilde{\mathbf{h}};k\rangle = B^k{}_l|\mathbf{h};l\rangle$. Only this sum is a trace of a well defined restriction of $q^{L_0-c/24}$ and does never contain any logarithmic parts. But the characters can: For example change of the basis $\{|h;0\rangle, |h;1\rangle\}$ to $\{|\tilde{h};0\rangle = |h;0\rangle + |h;1\rangle, |\tilde{h};0\rangle = -|h;0\rangle + |h;1\rangle\}$ yields

$$q^{L_0} = \frac{1}{2}\left(\begin{array}{cc} (1-2\pi i\tau)q^{L_{0;0}} + q^{L_{0;1}} & (1+2\pi i\tau)q^{L_{0;0}} - q^{L_{0;1}} \\ (1-2\pi i\tau)q^{L_{0;0}} - q^{L_{0;1}} & (1+2\pi i\tau)q^{L_{0;0}} + q^{L_{0;1}} \end{array}\right)\,.$$

We will return to this point in the last section, where an explicit realization of this is given for the CFT at $c = -2$. The generalization to larger Jordan cells is straightforward.

Since the characters of a CFT can be viewed as the zero-point-functions on a torus with modular parameter τ, they in general turn out to be certain modular functions whose Fourier expansions around $\tau = i\infty$ are just the q-series. One of the most powerful tools in CFT is the modular invariance of the partition function

$$Z(\tau,\bar\tau) = (q\bar q)^{\frac{c}{24}}\mathrm{tr}(q^{L_0}\bar q^{\bar L_0})\,. \tag{2.8}$$

The exciting result of J.L. Cardy [Car86], which was then mathematical rigorously proven by W. Nahm [Nah91], is that conformal invariance of a field theory on S^2 implies modular invariance of the partition function of the field theory on a torus. Unfortunately, the proof is strictly valid only for theories with diagonalizable L_0, but the result should also be valid for logarithmic CFTs (if the full Jordan cell is taken as basis for the HWRs), since the logarithmic behavior appears as a consequence of degeneracies of certain differential equations due to discrete and isolated values in the parameter space (essentially $\mathbf{h} \in \mathcal{C}^*$). Therefore, we expect that modular invariance can be extended to the logarithmic case in the same way as the logarithmic solutions of the differential equations are analytic continuations from the regular case. Since the partition function is a quadratic form in the characters, modular invariance puts severe restrictions on the modular behavior of the (generalized) characters. It will turn out that modular invariance uniquely determines a basis of HWSs within each Jordan block and therefore all characters.

We now fix some notations for the following. We will very often use the so called *elliptic functions* or *Jacobi-Riemann* Θ*-functions* which are modular forms of weight 1/2, defined as

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k}\,,\quad \lambda\in\mathbb{Z}/2\,,\ k\in\mathbb{N}/2\,. \tag{2.9}$$

We call λ the *index* and k the *modulus* of the Θ-function. The Θ-functions obey $\Theta_{\lambda,k} = \Theta_{-\lambda,k} = \Theta_{\lambda+2k,k}$, and $\Theta_{k,k}$ has, as power series in q, only even coefficients. We also need the *Dedekind* η*-function* which is defined as $\eta(\tau) = q^{1/24}\prod_{n\in\mathbb{N}}(1-q^n)$. The modular properties of these functions are for $\lambda, k \in \mathbb{Z}$

$$\begin{aligned}
\Theta_{\lambda,k}(-\tfrac{1}{\tau}) &= \sqrt{\tfrac{-i\tau}{2k}}\sum_{\lambda'=0}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}}\Theta_{\lambda',k}(\tau)\,, && (2.10)\\
\Theta_{\lambda,k}(\tau+1) &= e^{i\pi\frac{\lambda^2}{2k}}\Theta_{\lambda,k}(\tau)\,, && (2.11)\\
\eta(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\eta(\tau)\,, && (2.12)\\
\eta(\tau+1) &= e^{\pi i/12}\,\eta(\tau)\,. && (2.13)
\end{aligned}$$

The functions $\Lambda_{\lambda,k}(\tau) = \Theta_{\lambda,k}(\tau)/\eta(\tau)$ are then modular forms of weight zero to a particular main-congruence subgroup $\Gamma(N) \subset \mathrm{PSL}(2,\mathbb{Z})$, e.g. N is the least common multiple of $4k$ and 24 for $k \in \mathbb{Z}$.

As we have seen above, the characters for logarithmic CFTs are functions in the ring $\mathbb{Z}[[q]][\log q]$. Therefore we introduce the following additional functions:

$$(\partial\Theta)_{\lambda,k}(\tau) \propto \frac{\partial}{\partial\lambda}\Theta_{\lambda,k}(\tau) = \frac{2\pi i\tau}{k}\sum_{n\in\mathbb{Z}}(2kn+\lambda)q^{(2kn+\lambda)^2/4k}\,, \tag{2.14}$$

where we made explicit that new linear independent solutions of degenerate differential equations can be obtained by a formal derivation of the degenerate solution with respect to its parameter. As long as modular covariance is not concerned, there is no reason why τ could not appear as a factor. We introduce the so-called *affine* Θ-functions

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}}(2kn+\lambda)q^{(2kn+\lambda)^2/4k}\,, \tag{2.15}$$

which play an important rôle in the character formulas for the affine $\widehat{\mathfrak{su}(2)}$-algebra. They are odd, i.e. $(\partial\Theta)_{-\lambda,k} = -(\partial\Theta)_{\lambda,k}$. Moreover, per definitionem $(\partial\Theta)_{0,k} = (\partial\Theta)_{k,k} \equiv 0$. Their modular behavior is

$$\begin{aligned} (\partial\Theta)_{\lambda,k}(-\tfrac{1}{\tau}) &= (-i\tau)\sqrt{\tfrac{-i\tau}{2k}}\sum_{\lambda'=1}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}}(\partial\Theta)_{\lambda',k}(\tau)\,,\\ (\partial\Theta)_{\lambda,k}(\tau+1) &= e^{i\pi\frac{\lambda^2}{2k}}(\partial\Theta)_{\lambda,k}(\tau)\,. \end{aligned} \tag{2.16}$$

Since they are no longer modular forms of weight 1/2 under $S:\tau\mapsto -1/\tau$, we have to add further functions

$$(\nabla\Theta)_{\lambda,k}(\tau) = \frac{\log q}{2\pi i}\sum_{n\in\mathbb{Z}}(2kn+\lambda)q^{(2kn+\lambda)^2/4k} \tag{2.17}$$

in order to obtain a closed finite dimensional representation of the modular group. It is clear that S interchanges these two sets of functions, while $T:\tau\mapsto\tau+1$ transforms $(\nabla\Theta)_{\lambda,k}$ into $(\nabla\Theta)_{\lambda,k} + (\partial\Theta)_{\lambda,k}$. Therefore, the linear combination

$$(\partial\Theta)_{\lambda,k}(\tau)(\nabla\Theta)^*_{\lambda,k}(\bar\tau) - (\nabla\Theta)_{\lambda,k}(\tau)(\partial\Theta)^*_{\lambda,k}(\bar\tau) = (\tau-\bar\tau)|(\partial\Theta)_{\lambda,k}|^2$$

is modular covariant of weight 1/2!

Of course, the modular differential equation could be degenerate of higher degree, and one had to introduce generalizations $(\partial^n\Theta)_{\lambda,k}$ and $(\nabla^n\Theta)_{\lambda,k}$ (the expression $(\tau-\bar\tau)^n$ is modular covariant of weight $-2n$ for all $n\in\mathbb{Z}_+$). One can show [EhSk94] that regular rational theories with $c_{\mathit{eff}} \leq 1$ can only have one power $\eta(\tau)\eta(\bar\tau)$ in the denominator of the partition function. Regular means that the characters are modular forms. Now, the modular behavior of characters of logarithmic CFTs is almost the one of modular forms, except the possibility to expand into a power series in q. In particular, the asymptotic properties needed in the proof [EhSk94] are only affected in an analytic way by logarithmic corrections: In fact, although the modular differential equation makes only sense for particular isolated points in parameter space, $(c,\mathbf{h}_1,\mathbf{h}_2,\ldots)\in\bigoplus\mathcal{C}^*$, where the corresponding CFT is rational, it can be regarded as a differential equation depending on continuously variable parameters – once it has been written down. The characters of our theories in question are solutions of certain degenerate modular differential equations, obtained in a unique way by analytic continuation. Therefore, we conjecture that the result of [EhSk94] should also hold for logarithmic RCFTs. Thus, we should only be concerned with $n=1$ in our case.

3 Characters for the $c_{p,1}$ models

We already have seen that the $c_{p,1}$ models are logarithmic CFTs. V. Gurarie [Gur93] first derived the consequences of the existence of logarithmic fields. He explicitly considered the example of the degenerate Virasoro model at $c=-2$, i.e. the $c_{2,1}$ model. As explained above,

logarithmic CFTs always have *inequivalent* HWRs to the same HWV $\mathbf{h}$. A more detailed analysis of degenerate models shows that then always a further representation exists which has the generic null vector of the (regular) HWR on $|\mathbf{h}\rangle$ as HWS. This reflects the fact that many properties of the characters are only defined modulo $\mathbb{Z}$, since the HWR on the null vector of the representation $\mathfrak{H}_{|h\rangle}$ has the highest weight $h+k$ with $k \in \mathbb{Z}_+$. We are now going to derive the characters of these models. First, we show that the singlet models $\mathcal{W}(2, 2p-1)$ are not rational since the chiral symmetry algebra is too small for that.

3.1 Characters of the singlet algebras $\mathcal{W}(2, 2p-1)$

The additional primary field of the $\mathcal{W}(2, 2p-1)$-algebra is just the symmetric singlet of the $\mathfrak{su}(2)$ triplet of primary fields which generate the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$. One way to obtain the characters is to explicitly calculate the vacuum character and then get the others by modular transformations. From the embedding structure of Virasoro Verma modules for the values $c = c_{p,1}$ of the central charge [FeFu82, FeFu83, Fel89, F^{2}K89] we learn that the Virasoro character for the HWR on $|h_{2n+1,1}\rangle$, $n \in \mathbb{Z}_+$, is given by

$$\chi^{Vir}_{2n+1,1}(\tau) = \frac{q^{(1-c)/24}}{\eta(\tau)} \left(q^{h_{2n+1,1}} - q^{h_{-2n-1,1}}\right) . \tag{3.1}$$

Therefore [Flo93], the character of the $\mathcal{W}$-algebra vacuum representation is

$$\begin{aligned} \chi^{\mathcal{W}}_0(\tau) &= \sum_{n\in\mathbb{Z}_+} \chi^{Vir}_{2n+1,1}(\tau) & (3.2)\\ &= \frac{q^{(1-c)/24}}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{\frac{(2pn+p-1)^2}{4p}} , & (3.3) \end{aligned}$$

where we defined $\mathrm{sgn}(0) = 0$. It is convenient to rewrite the signum function as $\mathrm{sgn}(n+\frac{p-1}{2p})$. This character seems (up to the signum function) to be quite similar to the classical $\mathfrak{su}(2)$-Θ-function $\Theta_{p-1,p}(\tau, 0, 0)$ divided by η. Note, that the classical $\mathfrak{su}(2)$-Θ-functions $\Theta_{\lambda,k}(\tau, z, u)$, coincide for $z = u = 0$ with the elliptic functions defined in (2.9). They are the building stones for the characters of the $\widehat{\mathfrak{su}(2)}$ Kac-Moody-algebra. We therefore define

$$\Xi_{n,m}(\tau) = \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \mathrm{sgn}(k) q^{mk^2} . \tag{3.4}$$

But the modular transformation behavior is quite different from (2.10), while the presence of the signum function does not change the behavior under T, $\Xi_{n,m}(\tau+1) = \exp(i\pi\frac{n^2}{2m})\Xi_{n,m}(\tau)$. In order to get the behavior under S, we rewrite the functions $\Xi_{n,m}$ as linear combinations of $\Theta_{\lambda,k}$ functions. For this we introduce

$$\sigma(x, y) = \lim_{\varepsilon\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i y p^2}}{p + i\varepsilon^2} \left(e^{ipx} - e^{-ipx}\right) dp , \tag{3.5}$$

such that $\sigma(x, 0) = \mathrm{sgn}(x)$. In the following we omit the obvious limiting procedure. We find

$$\Xi_{n,m}(\tau) = \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \sigma(k, 0) q^{mk^2}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dp}{p} \left(e^{2\pi i k p} - e^{-2\pi i k p}\right) q^{mk^2} \qquad (3.6)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dp}{p} \left(\Theta_{n,m}(\tau,p,0) - \Theta_{n,m}(\tau,-p,0)\right) .
\end{aligned}
$$

Therefore, by linearity of the S-transformation, we can write

$$
\Xi_{n,m}(-\frac{1}{\tau}) = \tilde{\tilde{\Xi}}_{n,m}(\tau) = \sqrt{\frac{-i\tau}{2m}} \sum_{n' \bmod 2m} \sin(-2\pi\frac{nn'}{2m})\Xi_{n',m}(\tau) , \qquad (3.7)
$$

where $\tilde{\tilde{\Xi}}_{n,m}$ is given by

$$
\begin{aligned}
\tilde{\tilde{\Xi}}_{n,m} &= \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \sigma(k, -\frac{1}{2m\tau}) q^{mk^2}\\
&= \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \mathrm{erf}\left(\sqrt{\frac{-m\tau}{4\pi i}}k\right) q^{mk^2} . \qquad (3.8)
\end{aligned}
$$

Here, $\mathrm{erf}(x)$ denotes the usual Gauss error function up to normalization. To derive the last equality, one has to use the scaling invariance of the integral measure $\frac{dp}{p}$. Although the set of functions $\Xi_{n,m}$ and $\tilde{\tilde{\Xi}}_{n,m}$ closes under the S-transformation, they do not form a representation of the full modular group, since the $\tilde{\tilde{\Xi}}_{n,m}$ do not close under T. This means that they do not have a good power series expansion in q with integer coefficients and powers which differ by integers only. From this follows that the modular group forms an infinite dimensional representation by repeated action of T on $\tilde{\tilde{\Xi}}_{n,m}$. Therefore we conclude that the $\mathcal{W}(2, 2p-1)$-algebras do not yield RCFTs.

Similar to the case of the elliptic functions $\Theta_{\lambda,k}$, one may introduce additional variables which correspond to additional quantum numbers. For example we could write

$$
\Xi_{n,m}(\tau, z) = \sum_{k\in\mathbb{Z}+\frac{n}{2m}} \sigma(k, z) q^{mk^2} . \qquad (3.9)
$$

The variable z could belong to the eigenvalue of the additional element W_0 of the Cartan subalgebra, actually to its square, since only the latter can be determined. From the transformation behavior of the $\mathfrak{su}(2)$-Θ-functions [KaPe84] we get

$$
\begin{aligned}
\Xi_{n,m}(\tau+1, z) &= e^{\frac{\pi i n^2}{2m}} \Xi_{n,m}(\tau, z) , \qquad (3.10)\\
\Xi_{n,m}(-\frac{1}{\tau}, z\tau^2 - \frac{\tau}{2m}) &= \sqrt{\frac{-i\tau}{2m}} \sum_{n' \bmod 2m} \sin(-2\pi\frac{nn'}{2m})\Xi_{n',m}(\tau, z) . \qquad (3.11)
\end{aligned}
$$

Indeed, this set of functions forms a finite dimensional representation of the modular group. But the presence of an additional quantum number indicates that the chiral symmetry algebra is not yet maximally extended. Some remarks on this may be found in [FHW93, FrWe93].

3.2 Characters of the triplet algebras $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$

We now study the triplet algebras. Again, we construct $\mathcal{W}$-characters by summing up the Virasoro characters of degenerate representations whose highest weights differ by integers. In

addition, we have to take care of multiplicities coming from the $\mathfrak{su}(2)$ symmetry. Using the isomorphism between fields and Fourier modes which span the Hilbert space of the vacuum representation, one easily sees that the multiplicity of the Virasoro HWR on $|h_{2k+1,1}\rangle$ is $2k+1$. In particular, the multiplicity for $h_{3,1} = 2p-1$, the dimension of the additional primary fields, is 3 as it should be. The Virasoro characters are due to Feigin and Fuks [FeFu83]

$$\chi^{Vir}_{2k+1,1} = \frac{1}{\eta(q)}\left(q^{h_{2k+1,1}} - q^{h_{2k+1,-1}}\right)\,, \tag{3.12}$$

since there is precisely one singular vector in these representations. The vacuum representation of the $\mathcal{W}$-algebra is then the Hilbert space

$$\mathfrak{H}^{\mathcal{W}}_{|0\rangle} = \bigoplus_{k\in\mathbb{Z}_+}(2k+1)\mathfrak{H}^{Vir}_{|h_{2k+1,1}\rangle}\,. \tag{3.13}$$

Therefore, the vacuum character is

$$\begin{aligned}
\chi^{\mathcal{W}}_0 &= \sum_{k\in\mathbb{Z}_+}(2k+1)\chi^{Vir}_{2k+1,1} \\
&= \frac{q^{(1-c)/24}}{\eta(q)}\left(\sum_{k\geq 0}(2k+1)q^{h_{2k+1,1}} - \sum_{k\geq 0}(2k+1)q^{h_{-(2k+1),1}}\right) \\
&= \frac{q^{(1-c)/24}}{\eta(q)}\left(\sum_{k\geq 0}(2k+1)q^{h_{2k-1,1}} + \sum_{k\geq 1}(-2k+1)q^{h_{-2k+1,1}}\right) \\
&= \frac{q^{(1-p)^2/4p}}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k+1)q^{(1-(2k+1)p)^2-(1-p)^2)/4p} \\
&= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k+1)q^{(2pk+(p-1))^2/4p}\,.
\end{aligned} \tag{3.14}$$

This can be expressed in terms of Θ-functions and affine Θ-functions as

$$\chi^{\mathcal{W}}_0 = \frac{1}{p\eta(\tau)}\left((\partial\Theta)_{p-1,p}(\tau) + \Theta_{p-1,p}(\tau)\right)\,. \tag{3.15}$$

But now we are in trouble here, since only the functions $\Lambda_{\lambda,k} = \Theta_{\lambda,k}/\eta$ are modular forms of weight zero, while the terms $(\partial\Lambda)_{\lambda,k} = (\partial\Theta)_{\lambda,k}/\eta$ have the modular weight 1.

Let us consider the modular transformation behavior of $(\partial\Lambda)_{\lambda,k}$ under S and T. From (2.10) we get the relations

$$\begin{aligned}
(\partial\Lambda)_{\lambda,k}(\tau+1) &= \exp\left(2\pi i\left(\frac{\lambda^2}{4k} - \frac{1}{24}\right)\right)(\partial\Lambda)_{\lambda,k}\,, && (3.16)\\
(\partial\Lambda)_{\lambda,k}(-\frac{1}{\tau}) &= (-i\tau)\sqrt{\frac{2}{k}}\sum_{1\leq\lambda'\leq k-1}\sin\left(\frac{\pi\lambda\lambda'}{k}\right)(\partial\Lambda)_{\lambda',k}\,. && (3.17)
\end{aligned}$$

Note the occurrence of a term τ, which cannot be written as a power series in q. We define $(\nabla\Lambda)_{\lambda,k} \equiv -\tau(\partial\Lambda)_{\lambda,k}$, which have the modular properties

$$\begin{aligned}
(\nabla\Lambda)_{\lambda,k}(\tau+1) &= \exp\left(2\pi i\left(\frac{\lambda^2}{4k} - \frac{1}{24}\right)\right)\left((\nabla\Lambda)_{\lambda,k} - (\partial\Lambda)_{\lambda,k}\right)\,, && (3.18)\\
(\nabla\Lambda)_{\lambda,k}(-\frac{1}{\tau}) &= -i\sqrt{\frac{2}{k}}\sum_{1\leq\lambda'\leq k-1}\sin\left(\frac{\pi\lambda\lambda'}{k}\right)(\partial\Lambda)_{\lambda',k}\,. && (3.19)
\end{aligned}$$

It is remarkable, that the T-transformation is no longer diagonal. In some cases the h-values of the allowed HWRs are explicitly known. These are $\mathcal{W}(2,3,3,3)$ at $c=-2$, with the only possible highest weights $h\in\{-1/8,0,3/8,1\}$, and $\mathcal{W}(2,5,5,5)$ at $c=-7$, which has HWRs for $h\in\{-1/3,-1/4,0,5/12,1,7/4\}$ only. With these data one can solve the modular differential equation to find the characters. The result is up to base changes the same.

We would like to note that one can formally read off the possible representations from the conformal grid of minimal models in the following way: The possible h-values of a minimal model with $c=c_{p,q}$ are given by $h_{r,s}=\frac{(pr-qs)^2-(p-q)^2}{4pq}$ with $1\leq r<q$ and $1\leq s<p$. One obtains the h-values for a $c_{p,1}$-model including all inequivalent representations to the same highest weight from the conformal grid of $c_{3p,3}$.

For simplicity, we concentrate now on the case $c=-2$, i.e. $p=2$. We first assume the usual form of the characters,

$$\chi_i = q^{h_i-c/24}\sum_{l=0}^{\infty} b_{i,l}q^l\,, \tag{3.20}$$

where h_i is given by $h_{1,i}=\frac{i^2-2ip+2p-1}{4p}$. Solving the modular differential equation yields up to multiplicative prefactors the characters

$$\begin{array}{rcl}\chi_1 &=& A\Lambda_{1,2}+B(\partial\Lambda)_{1,2}\,,\\ \chi_2 &=& \Lambda_{0,2}\,,\\ \chi_3 &=& A'\Lambda_{1,2}+B'(\partial\Lambda)_{1,2}\,,\\ \chi_4 &=& \Lambda_{2,2}\,,\\ \chi_5 &=& \frac{1}{2}\Lambda_{1,2}-\frac{1}{2}(\partial\Lambda)_{1,2}\,.\end{array} \tag{3.21}$$

Therefore, χ_1, χ_3 and χ_5 are linear dependent. If χ_1 is supposed to belong to the vacuum representation, its coefficient to q must vanish, i.e. $b_{1,1}=0$. This forces $A=B=1/2$, if one also requires $b_{1,0}=1$.

We now need one further, linear independent solution. We make the ansatz

$$\tilde{\chi}_3 = \log(q)q^{1/12}\sum_{l=0}^{\infty}\tilde{b}_{3,l}q^l\,. \tag{3.22}$$

Inserting this into the modular differential equation, we get

$$\tilde{\chi}_3 = (\nabla\Lambda)_{1,2}\,, \tag{3.23}$$

where from now on we define the characters as functions in q, i.e. $(\nabla\Lambda)_{\lambda,k}\equiv-\frac{\log(q)}{2\pi i}(\partial\Lambda)_{\lambda,k}$. Indeed, our result is exactly the same as what we got from the explicit calculation of the vacuum character and its S-transformation. Replacing χ_3 by $\tilde{\chi}_3$, we obtain the S-matrix:

$$S=\begin{pmatrix} 0 & \frac{1}{4} & \frac{i}{2} & -\frac{1}{4} & 0\\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1\\ -i & 0 & 0 & 0 & i\\ -1 & \frac{1}{2} & 0 & \frac{1}{2} & -1\\ 0 & \frac{1}{4} & -\frac{i}{2} & -\frac{1}{4} & 0\end{pmatrix}. \tag{3.24}$$

This matrix has, as it should be, $\det(S)=1$, but is neither symmetric nor real nor unitary. Nonetheless is satisfies $S^2=\mathbb{1}$. Most disturbing, it has no row/column with only non vanishing entries. We collect our intermediate results.

Proposition 2. *Let $p \in \mathbb{N}$. Then there exists at $c = c_{p,1} = 13 - 6(p + p^{-1})$ a $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$. There are precisely $3p-1$ HWRs with highest weights $h_{1,s}, 1 \leq s \leq 3p-1$. Of them $2\times(p-1)$ HWRs have pairwise identical highest weights, further $p-1$ highest weights differ from these pairs by positive integers which are the levels of the singular vectors. A basis for the characters is given by (η^{-1} times) the functions $\{\Theta_{\lambda,p}, (\partial\Theta)_{\mu,p}, (\nabla\Theta)_{\mu,p} | 0 \leq \lambda, \mu \leq (2p-1), \mu \neq 0, p\}$.*

We already noted that the functions $(\nabla\Theta)_{\mu,p}$ lead to a non diagonal T-matrix. It decomposes into blocks similar to Jordan cells, but which also mix characters whose corresponding highest weights differ by integers. For our example the T-matrix is (in the same basis)

$$T = \begin{pmatrix} \exp(\pi i/6) & & & & \\ & \exp(-\pi i/12) & & & \\ -\exp(\pi i/6) & 0 & \exp(\pi i/6) & 0 & \exp(\pi i/6) \\ & & & \exp(11\pi i/12) & \\ & & & & \exp(13\pi i/6) \end{pmatrix}. \qquad (3.25)$$

Nonetheless, this matrix satisfies together with the S-matrix (3.24) the relation $(ST)^3 = 1\!\!1$. This is very important in order to have modular invariance of the CFT. It is easy to see that the statement is true for the general case of proposition 2.

But what are the “physical” characters? This question will be answered by enforcing modular invariance of the partition function. From our discussion of the modular properties of the characters we know that the following expression is modular invariant:

$$Z_{log}[p] = \alpha \sum_{\lambda=0}^{2p-1} |\Theta_{\lambda,p}|^2 + \beta \sum_{\substack{\mu=1 \\ \mu\neq p}}^{2p-1} \frac{1}{2}\left((\partial\Theta)_{\mu,p}(\nabla\Theta)^*_{\mu,p} + (\nabla\Theta)_{\mu,p}(\partial\Theta)^*_{\mu,p}\right), \qquad (3.26)$$

where α, β are yet free constants. The partition function will only be physical relevant, i.e. with integer coefficients only, if $\alpha, \beta \in \mathbb{Z}/2$.

According to proposition 2 we introduce general linear combinations for the characters of each triplet of degenerate HWRs $(h, h, h+k)$. There is only one such triplet, $(0, 0, 1)$, in our example of $p = 2$. Then the ansatz is

$$\begin{aligned} \chi_0 &= \Theta_{0,2}/\eta\,, \\ \chi_2 &= \Theta_{2,2}/\eta\,, \\ \chi_1^0 &= (a_0\Theta_{1,2} + b_0(\partial\Theta)_{1,2} + c_0(\nabla\Theta)_{1,2})/\eta\,, \\ \chi_1^+ &= (a_+\Theta_{1,2} + b_+(\partial\Theta)_{1,2} + c_+(\nabla\Theta)_{1,2})/\eta\,, \\ \chi_1^- &= (a_-\Theta_{1,2} + b_-(\partial\Theta)_{1,2} + c_-(\nabla\Theta)_{1,2})/\eta\,. \end{aligned}$$

One sees immediately that a_i, b_i have to be real and c_i imaginary to obtain a (possibly) physical relevant partition function. Putting $a_0 \neq 0$ gives a contradiction. With $a_0 = 0$ one finds the solution $a_+ = a_- = 1$, $b_- = -b_+$ and $c_+ = -c_- = \pm ib_+$. Then also $\pm c_0 = -\sqrt{2}ib_+$ and $\pm b_0 = \sqrt{2}b_+$ are fixed up to a common sign. We see that logarithmic terms occur in *all* HWRs of the triplet, since otherwise we cannot cancel terms not allowed in (3.26). Moreover, the S-matrix now gets a block structure, since one $\mathcal{W}$-family, the one with the unphysical multiplicity $\sqrt{2}$, decouples from the remaining ones. It is remarkable that the four remaining HWRs obey a good fusion algebra under themselves, if one applies the Verlinde formula to

the corresponding block of the S-matrix, which is (including the free signs)

$$S = \begin{pmatrix} \mp 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & \pm 1/2 & -1/2 & \mp 1/2 \\ 0 & 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 1/2 & \mp 1/2 & -1/2 & \pm 1/2 \end{pmatrix} . \tag{3.27}$$

Here, the third row/column corresponds to the vacuum representation with χ_1^+ and the character vector is $(\chi_1^0, \chi_0, \chi_1^+, \chi_2, \chi_1^-)^t$. This new S-matrix is symmetric and unitary. The remaining constants are fixed by the T-matrix. It follows $b_+ = \pm 1$, where a sign change corresponds to an exchange of the third and fifth row/column. Choosing $b_+ = 1$ selects χ_1^+ as vacuum character, whose non logarithmic part is – up to an overall multiplicity of 2 – identical to our first result. The T-matrix then reads

$$T = \begin{pmatrix} (1-\frac{i}{4})\exp(\pi i/6) & 0 & \frac{i}{4}\exp(\pi i/6) & -\frac{i}{2\sqrt{2}}\exp(\pi i/6) & 0 \\ 0 & \exp(\pi i 11/12) & 0 & 0 & 0 \\ \frac{i}{4}\exp(\pi i/6) & 0 & (1-\frac{i}{4})\exp(\pi i/6) & \frac{i}{2\sqrt{2}}\exp(\pi i/6) & 0 \\ \frac{i}{2\sqrt{2}}\exp(\pi i/6) & 0 & -\frac{i}{2\sqrt{2}}\exp(\pi i/6) & (1+\frac{i}{2})\exp(\pi i/6) & 0 \\ 0 & 0 & 0 & 0 & \exp(-\pi i/12) \end{pmatrix} . \tag{3.28}$$

The enlarged multiplicity cannot be avoided, since otherwise the partition function (3.26) would have rational non integer coefficients. In the general case, the enlarged multiplicity is just p, and stems from the degenerate solutions of chiral vertex operators of same conformal dimension (as explained in section 1). In general we now have

PROPOSITION 3. *Let $p \in \mathbb{N}, p \geq 2$. Then there are the following HWRs and characters for the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$-algebra: A unique HWR to the highest weight of minimal energy, $h_{min} = h_{1,p} = h_{0,0} = -(p-1)^2/4p$, with character $\chi_p = \Theta_{0,p}/\eta$, one HWR to $h_{1,2p} = h_{1,0}$ with character $\chi_0 = \Theta_{p,p}/\eta$, both do not contain any null states. Moreover, triplets of HWRs to $(h_{1,s} = h_{1,2p-s}, h_{1,2p+s})$ with inequivalent [ii] characters $(\chi_s^+, \chi_s^0, \chi_s^-)$ where $\chi_s^\pm = (\Theta_{p-s,p} \pm (\partial\Theta)_{p-s,p} \pm i(\nabla\Theta)_{p-s,p})/\eta$ and $\chi_s^0 = \sqrt{2}((\partial\Theta)_{p-s,p} - i(\nabla\Theta)_{p-s,p})/\eta$. The "diagonal" partition function is then*

$$Z_{log}[p] = \frac{1}{\eta\bar{\eta}} \left(|\chi_0|^2 + |\chi_p|^2 + \sum_{1 \leq s \leq p-1} \left(|\chi_s^0|^2 + \chi_s^+(\chi_s^-)^* + \chi_s^-(\chi_s^+)^* \right) \right) . \tag{3.29}$$

Noting that $(\partial\Theta)_{0,p} = (\nabla\Theta)_{p,p} \equiv 0$, one can write all characters in the same form. The resulting S-matrix has block structure. $p-1$ logarithmic representations with characters $\chi_s^0, 1 \leq s \leq p-1$ decouple from the remaining $2(p-1)+2 = 2p$ "regular" representations. For the latter we already derived abstract fusion rules in section 1. We remark that the quantum dimensions of the decoupling representations all vanish. This and the block structure of the S-matrix show that the Verlinde formula is no longer valid. We only can calculate the fusion rules for the $2(p-1)$ regular representations, if we use the corresponding block of the S-matrix. Nonetheless, the fusion rules calculated from the "regular" block of the S-matrix are

[ii] Note, that the singular vector of the HWR to $h_{1,s}$ has level $h_{1,-s} = h_{1,2p+s}$, but the singular vector of the HWR to $h_{1,2p-s}$ has level $h_{1,s-2p} = h_{1,4p-s}$.

well behaved. In the last section we will discuss their physical relevance. For our example we obtain from the Verlinde formula, applied to the 4×4 block,

$$\begin{array}{ccccc} [-\frac{1}{8}] & \times & [-\frac{1}{8}] & = & [0]\,, \\ [-\frac{1}{8}] & \times & [0] & = & [-\frac{1}{8}]\,, \\ [-\frac{1}{8}] & \times & [\frac{3}{8}] & = & [1]\,, \\ [-\frac{1}{8}] & \times & [1] & = & [\frac{3}{8}]\,, \\ [0] & \times & [0] & = & [0]\,, \end{array} \qquad \begin{array}{ccccc} [0] & \times & [\frac{3}{8}] & = & [\frac{3}{8}]\,, \\ [0] & \times & [1] & = & [1]\,, \\ [\frac{3}{8}] & \times & [\frac{3}{8}] & = & [0]\,, \\ [\frac{3}{8}] & \times & [1] & = & [-\frac{1}{8}]\,, \\ [1] & \times & [1] & = & [0]\,, \\ \text{and } [\tilde{0}] & \times & [\tilde{0}] & = & [\tilde{0}]\,. \end{array} \tag{3.30}$$

The second representation with $h = 0$, $[\tilde{0}]$, completely decouples from the other representations. This is similar to a well known phenomenon in q-algebras, where representations with vanishing quantum dimensions are invisible for the other representations. This also explains the unphysical multiplicity $\sqrt{2}$ for characters of such representations. Actually, the underlying q-algebra structure admits to representations whose quantum dimensions add up to zero. Thus, the corresponding $\mathcal{W}$-algebra representations are degenerated. For example, we have two representations $[\tilde{0}_\pm]$ with characters $\chi^+_{1,0} = -\chi^-_{1,0} = \chi_{1,0}/\sqrt{2}$.

The case $p = 1$ is trivial, there are no logarithmic representations. It is just $\widehat{\mathfrak{su}(2)}$, the simplest non-abelian infinite dimensional Lie algebra $A_1^{(1)}$, with $c = 1$. In particular, $Z_{\rm log}[1] = Z[1]$, where $Z[x]$ denotes the standard Gaussian U(1) partition function for a free field compactified with radius $R = \sqrt{(x/2)}$, usually denoted $Z(R)$. This means that our logarithmic CFT reduces to the Gauss model at the multi-critical point of radius $1/\sqrt{2}$. But R. Dijkgraaf and E. & H. Verlinde [DV^{2}88] have proven that there are *no* marginal deformations, which can lead out of the known moduli space of $c = 1$ CFTs. There is one field of marginal dimension, $\phi_{2,p-1}$ with $h_{2,p-1} = 1$, which belongs to the (extended) conformal grid of section 1. Since the first label is even, it has vanishing self coupling, which is necessary for a marginal operator to be integrable. But this field does not exist for $p = 1$, since all fields $\phi_{r,s}$ with $r = 0$ or $s = 0$ decouple completely from the physical Hilbert space due to annihilation by the BRST operator. Thus, we indeed cannot go from the moduli space of regular $c = 1$ CFTs to the logarithmic CFTs with $c_{\mathit{eff}} = 1$ via marginal deformations. If we finally note that the partition function of proposition 3 also allows non diagonal decompositions, we have

Proposition 4. *The moduli space of logarithmic CFTs with $c_{\mathit{eff}} = 1$ is generic one dimensional and not connected to the moduli space of regular $c = 1$ CFTs. The partition function of a logarithmic CFT is for $(p, q) = 1$ given by*

$$Z_{log}[p/q] = \frac{1}{\eta\bar{\eta}}\left(|\chi_0|^2 + |\chi_{pq}|^2 + \sum_{1\leq s\leq pq-1}\left(\chi^0_s(\chi^0_{s'})^* + \chi^+_s(\chi^-_{s'})^* + \chi^-_s(\chi^+_{s'})^*\right)\right)\,, \tag{3.31}$$

where $s = pn - qm$ mod $2pq$ implies $s' = pn + qm$ mod $2pq$.

The connected part of the moduli space of $c = 1$ theories has an exact copy of logarithmic theories in the following manner: First, one writes

$$Z_{log}[x] = \left(1 + \frac{2x^2}{\pi i}\frac{\partial}{\partial x}\right) Z[x]\,, \tag{3.32}$$

which by the way defines $Z_{log}[x]$ for arbitrary, not necessarily rational x. In the same way we obtain the partition function of the $\mathbb{Z}_2$-orbifolds of the logarithmic theories by applying

$(1+\frac{2x^2}{\pi i}\partial_x)$ to $Z_{orb}[x]$,

$$Z_{log,orb}[x] = \left[\left(1+\frac{2x^2}{\pi i}\frac{\partial}{\partial x}\right)Z[x] + \left(1+\frac{2y^2}{\pi i}\frac{\partial}{\partial y}\right)Z[y]\Big|_{y=4} - Z[1]\right]/2\,. \tag{3.33}$$

The corresponding $\mathcal{W}$-algebras, which exist at points of enhanced symmetry analogous to the regular case, are the following: To $Z_{log}[p], p\in\mathbb{N}$ belongs a $\mathcal{W}(2,(2p-1)^{\otimes 3})$, whose $\mathbb{Z}_2$-orbifold contains a $\mathcal{W}(2,6p-2)$, the $\mathbb{Z}_2$-orbifold of $\mathcal{W}(2,2p-1)$ where the singlet field is given by $W=W_0+W_++W_-$ and the orbifold is obtained by identifying W with $-W$. Since the structure constant C_{WW}^W does not vanish for the triplet, the $\mathbb{Z}_2$-orbifold of the triplet should be given by the identifications $W_0\leftrightarrow -W_0$, $W_+\leftrightarrow -W_-$, and $W_-\leftrightarrow -W_+$ such that one field, e.g. $\tilde{W}=W_+-W_-$ survives. The orbifold would then be a $\mathcal{W}(2,2p-1,6p-2)$. If p is a complete square, $p=n^2$, these algebras can be extended by a field of dimension $h_{2n+1,1}=p(n^2+n)-n=n^4+n^3-n$. In the same manner one can write down the logarithmic analogs of the three exceptional $c=1$ partition functions. Setting $D_x=\frac{2x^2}{\pi i}\partial_x$, the exceptional logarithmic partition functions simply read

$$Z_{log,E_6} = \frac{1}{2}\left(\sum_{x\in\{4,9,9\}}(1+D_x)Z[x]-Z[1]\right)\,, \tag{3.34}$$

$$Z_{log,E_7} = \frac{1}{2}\left(\sum_{x\in\{4,9,16\}}(1+D_x)Z[x]-Z[1]\right)\,, \tag{3.35}$$

$$Z_{log,E_8} = \frac{1}{2}\left(\sum_{x\in\{4,9,25\}}(1+D_x)Z[x]-Z[1]\right)\,. \tag{3.36}$$

In this way, the full $c=1$ moduli space is recovered in the "logarithmic" regime. There are no other linear combinations possible, since the non-logarithmic part of the partition function has to satisfy the usual requirements to be physical relevant, which only yield the known $c=1$ solutions.

Of course, there could be higher powers of logarithmic terms. All expressions of the form $(\sum_{n\in\mathbb{Z}_+}a_nD_x^n)Z[x]$ are modular invariant. Fortunately, as mentioned above, this presumably cannot happen for theories with $c_{eff}\leq 1$ (see also [Flo94PhD]).

We conclude with a remark on $N=1$ supersymmetric theories. The explicit known examples [BEH²92, EH²93] as well as the general results on the modular properties of characters make it clear that $N=1$ CFTs will have the same structure. One finds again logarithmic theories (with $c_{eff}=3/2$), which have a completely analogous representation theory. This analogy extends the similarity of the representation theory of the already known $N=0,1$ RCFTs [Flo93]. But as already observed in other cases, such results do not extend to $N=2$, since there no rational like structure can be found (for examples see [Blu93]). It remains the conjecture that for $N=2$ rationality of a CFT implies its unitarity.

4 Discussion: Polymers

Finally, we would like to discuss the consequences of our results in relation to the theoretical understanding of two dimensional polymers. In an early work [DuSa87], B. Duplantier and H. Saleur derived that the full partition function of two dimensional polymers in the dense

phase is trivially zero. But they also showed that the partition function of a single self-avoiding polygon (a dense loop polymer) homotopic to zero on a torus is given by

$$\tilde{X} = -\frac{1}{4\pi}\eta^2(q)\eta^2(\bar{q})\log(q\bar{q}) = (D_x Z[x])|_{x=1/2} , \tag{4.1}$$

which is the modular invariant part of of the summed number of configurations. Surprisingly, this is precisely the logarithmic part of our $c = -2$ partition function.

In a later work [Sal92] H. Saleur studied the polymer problem again and found a hidden $N = 2$ supersymmetry by modeling it via a η-ξ system. This model has different sectors which combine into two modular invariant partition functions, one for even number of non-contractable polymers, which is $Z_{even} = \frac{1}{2}Z[2]$, and one for odd number of non-contractable polymers, which is $Z_{odd} = Z[8] - \frac{1}{2}Z[2]$. The latter corresponds to a so called twisted sector of the η-ξ system with $\mathbb{Z}_4$ symmetry. We find now the following: Z_{even} equals the regular part of our partition function of the $c = -2$ model, Z_{odd} is identical to the partition function of the $c = -2$ model with twisted bosons, i.e. the bosonic fields have now half-integer Fourier modes. One obtains the partition function from the modular properties of the Jacobi-Riemann elliptic functions $\Theta_{\lambda,p}$ with half-integer index λ. In fact, the twisted $\mathcal{W}(2,3,3,3)$-model allows only two HWRs which have $h = -5/32$ and $h = 3/32$. They have the characters $\chi_{1/2} = \Theta_{\frac{1}{2},2}/\eta$ and $\chi_{3/2} = \Theta_{\frac{3}{2},2}/\eta$ respective. The resulting partition function is just Z_{odd}.

As a result, the modular invariant partition function of our $\mathcal{W}(2,3,3,3)$-model at $c = -2$ naturally contains *all* different partition functions of two-dimensional dense polymers, *including* the logarithmic part which comes from polymers homotopic to zero. The hidden $N = 2$ supersymmetry in form of a η-ξ system still is visible in the behavior of the screening currents: Q and the current $\tilde{J}$ corresponding to $\tilde{Q}$ create states similar but not equal to a usual η-ξ system.

The difference between our approach and the one with $N = 2$ supersymmetry is that the latter does not detect the states corresponding to the logarithmic operators. But as shown in [Gur93], these operators cannot be avoided, if the OPE and the construction of conformal blocks should remain consistent. Without these additional states one ends up with a partition function which is equal to an ordinary $c = 1$ Gaussian model. Only the h-values change due to the shift of the central charge to $c = -2$. Certainly, the structure of the CFTs at $c = 1$ and $c = -2$ is different. This is not visible in the supersymmetric approach, presumably because the corresponding chiral algebra is not maximal. The additional states are naturally accounted for in our model. Surprisingly this just leads to an additional term in the partition function which represents the polymers homotopic to zero.

One possible way to find the missing states in the η-ξ system is the observation that the naive L_0 from $T(z) =: \eta\partial\xi :$ is diagonal. To recover the correct Jordan cell structure, which only is present in the Neveu-Schwarz sector, one can modify L_0 in the way

$$L_0 = \sum_{m=1}^{\infty} m(\xi_{-m}\eta_m + \eta_{-m}\xi_m) + \eta_0 , \tag{4.2}$$

which does not change any commutators, since $\{\xi_m, \eta_n\} = \delta_{m+n,0}$ and thus $[L_m, \eta_n] = n\eta_{m+n}$. By this, we correctly obtain $L_0(\xi_0|0\rangle) = |0\rangle$, such that the two states $|0;0\rangle = |0\rangle$ and $|0;1\rangle = \xi_0|0\rangle \neq 0$ span the Jordan cell. This also justifies a posteriori our definition of characters for non-diagonal L_0, as given in the second section. Actually, the form given

above naturally matches the definition of L_0 as matrix, given in section 2, where the non-diagonal part only contributes to the states on $|0;1\rangle$. But now, (4.2) gives L_0 in such a way that taking traces is still well defined, and thus characters can be defined as usual. This correction of L_0 is very reminiscent of a similar "explicit" visibility of the related quantum group structure in the CFT of Liouville theory [GoSi91], where also L_0 is non-diagonal.

Let us finally discuss the operator algebra. In [Sal92], the operators of the $c=-2$ CFT have been identified with the L-leg operators in the following way: In the antiperiodic (Ramond) sector, we have fields of conformal weights $h_{1,2+2l}$ which correspond to $4l$-leg operators. Their fermion number is $F \equiv l \bmod 2$. Our model collects these operators into two $\mathcal{W}$-conformal families, $[-\frac{1}{8}]$ with $F=0$, and $[\frac{3}{8}]$ with $F=1$. Due to the fermion number, the first HWR is a singlet, the second a doublet representation. The periodic (Neveu-Schwarz) sector contains fields of conformal weights $h_{1,3+4k}$ and $h_{1,5+4k}$ which correspond to the $(4l+2)$-leg operators for $l=2k$ and $l=2k+1$ respectively. Again, our model collects these operators into two families, $[0]$ for l even, and $[1]$ for l odd. For the sake of completeness, we just mention that the so called $\mathbb{Z}_4$ sector in [Sal92] coincides with the twisted sector of our model, which splits into the two families $[-\frac{3}{32}]$ and $[\frac{5}{32}]$ which correspond to the $(4l+1)$-leg operators and the $(4l+3)$-leg operators, where the first family again collects operators for l even, the second collects operators for l odd. In all families, the fields are descendants of the one with the lowest value L of legs. Thus, we obtain the following splitting of the operators modulo 8:

$$\begin{array}{rll} \mathrm{Ramond}: & L\equiv 0(8)\,, & \Phi_L\in[-\frac{1}{8}]\,,\\ & L\equiv 4(8)\,, & \Phi_L\in[\frac{3}{8}]\,,\\ \mathrm{Neveu-Schwarz}: & L\equiv 2(8)\,, & \Phi_L\in[0]\,,\\ & L\equiv 6(8)\,, & \Phi_L\in[1]\,,\\ \mathbb{Z}_4\ \mathrm{sector}: & L\equiv 1,3(8)\,, & \Phi_L\in[-\frac{3}{32}]\,,\\ & L\equiv 5,7(8)\,, & \Phi_L\in[\frac{5}{32}]\,. \end{array} \tag{4.3}$$

Due to the more complicated modular transformation behavior of the characters in the $\mathbb{Z}_4$ sector, which involves alternating Θ-functions $\tilde{\Theta}_{\lambda,k}=\sum_{n\in\mathbb{Z}}(-)^n q^{(4kn+\lambda)^2/4k}$, and therefore a second set of characters $\tilde{\chi}(\tau)\equiv\chi(\tau+1)$, the splitting modulo 8 can be made rigorous by considering $(\chi\pm\tilde{\chi})/2$.

The fusion rules (3.30) respect the sector structure derived by H. Saleur [Sal92]. The $4l$-leg operators build the partition function which (on the lattice put on a torus) sums over dense coverings by an even number of non-contractable polymers that cross both periods in the total an odd number of times. Of particular interest are the representations for the $(4l+2)$-leg operators. Their characters combine to the part of the partition function which (on the lattice put on a torus) sums over dense coverings by an even number of non-contractable polymers that cross one period an odd number of times. But they also yield the part of the partition function which is the number of configurations of a single contractable densely covering polymer. Actually, the naive partition function for doubly periodic boundary conditions vanishes, but its derivative is precisely this logarithmic part of our full partition function. The twisted sector partition function (on the lattice put on a torus) sums over dense coverings by an odd number of non-contractable polymers.

We see, that the Neveu-Schwarz sector also contains the contribution of one dense, contractable loop. This contribution stems from the Jordan cell structure, i.e. the fact that application of L_0 produces new states. In the Temperley-Lieb picture of polymer configu-

rations it is natural to use a single dense contractable polymer as a possible groundstate from which non-contractable configurations can be created by action of the Termperley-Lieb operators. In fact, the BRST operator

$$Q_{\mathrm{BRST}} = \oint \frac{dz}{2i\pi} \eta(z) \tag{4.4}$$

is the polymer creating operator, which may as well act on the sector without polymers as on the sector with exactly one contractable polymer as groundstates. More than one contractable polymer always yields zero due to the special property of the corresponding Temperley-Lieb algebra that $e^2 = \delta e$ with $\delta = 0$. The BRST property $Q^2_{\mathrm{BRST}} = 0$ remains valid in the sector with one single contractable polymer.

Physically, the three HWRs [0], [1], and $[\tilde{0}]$ correspond to three different ways of counting states build on groundstates with zero or one contractable dense loop. The actual combinations are selected by modular invariance of the partition function. [0] and [1] count states from both groundstates with and without a sign for states with contractable loop. $[\tilde{0}]$ may be interpreted as the HWR to the density field.

Since it is believed that the higher members of the $c_{1,p}$ series describe multi-critical polymers, we conjecture that the corresponding $\mathcal{W}(2, (2p-1)^{\otimes 3})$ models are the right candidates for these physical systems. In fact, the Jordan cell structure of L_0 is a common feature of all members of the $c_{1,p}$ series, which is naturally incorporated in these multiplet $\mathcal{W}$-algebras.

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